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Research Article

Isoperimetric Certificates for Flocking: Weighted Cheeger Bounds on Cohesion, Tracking, and Fragmentation in State-Dependent Networks

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Abstract

The continuous-time flocking law of Olfati-Saber is, in its alignment term, a flow generated by a state-dependent weighted graph Laplacian. We exploit this identification to bring the weighted Cheeger constants and Neumann/Dirichlet isoperimetric inequalities of Chung and Oden to bear on the analysis of second-order multi-agent dynamics. The control literature typically certifies flocking through direct spectral quantities—the algebraic connectivity (Fiedler value) of the interaction Laplacian. We instead derive cut-based certificates: (i) a Neumann isoperimetric lower bound on the rate of velocity (heading) consensus; (ii) a Dirichlet isoperimetric lower bound on the rate at which the flock locks onto a navigational target, recovering and refining grounded-Laplacian leader-follower estimates; and (iii) an isoperimetric fragmentation criterion that ties imminent splitting of the swarm to the collapse of the weighted Cheeger constant along a sparse cut. Because a cut is a local quantity, these certificates are monitorable without a global eigensolve; we make this operational with an analysis of computational complexity, distributed implementation, and robustness to measurement noise, and we characterize the regimes in which each bound is tight. In the mean-field limit we conjecture, and pose as an open problem, how the discrete weighted Cheeger constant converges to a continuum Cheeger constant, so that the discrete certificates would appear as consistent discretizations of the Riemannian Cheeger-Buser inequality. Finally we propose an isoperimetric connectivity-maintenance controller that regulates the weighted Cheeger constant directly, and contrast it with algebraic-connectivity maximization. An expanded numerical suite (varying agent number, density regime, and informed fraction) confirms every certificate and quantifies its slack.

Introduction

Collective motion in biological and engineered systems—bird flocks, fish schools, and swarms of autonomous vehicles—has been modeled by local interaction rules at least since the work of Reynolds and Vicsek et al. [1,2]. A control-theoretic foundation was given by Olfati-Saber [3], who cast flocking as a feedback problem and proposed a now-standard second-order law in which each agent's acceleration combines alignment (velocity matching), separation (collision avoidance), and navigational feedback to a moving objective. Convergence and stability of such laws are classically analyzed through algebraic graph theory: the spectral properties of the interaction (graph) Laplacian govern the rate of consensus and the cohesion of the group [4-7]. In particular, the algebraic connectivity or Fiedler value [8]—the smallest nonzero Laplacian

eigenvalue—sets the slowest decay mode of disagreement and has become the central object both in convergence analysis and in connectivity-control design [9-13]. This quantity is written λ_2 in the one-indexed convention of Fiedler and the control literature (where $0 = \lambda_1 \leq \lambda_2 \leq \dots$) and λ_1 in the zero-indexed convention of spectral graph theory (where $0 = \lambda_0 \leq \lambda_1 \leq \dots$) that we adopt throughout; the two symbols denote the same eigenvalue.

There is a parallel and largely separate tradition in spectral graph theory. The Cheeger constant—an isoperimetric (cut) quantity—bounds the spectral gap of the Laplacian from below, a discrete analogue of Cheeger's Riemannian inequality [14] and its Buser reverse [15]. Chung and Oden [16] developed the weighted theory, introducing weighted Cheeger constants and isoperimetric inequalities for both Neumann and Dirichlet boundary conditions. This

machinery is heavily used in graph partitioning, spectral clustering, and Markov-chain mixing [17,18], but it is rarely the instrument of choice in multi-agent control, where λ_2 is computed or estimated directly.

The present paper bridges the two. Our starting observation is elementary but consequential: the alignment term of the flocking law is exactly the action of a weighted graph Laplacian, with edge weights that depend on the agents' configuration. The isoperimetric apparatus of [16] therefore applies, and yields certificates that are geometric rather than spectral—expressed through cuts that can be monitored locally rather than through global eigenvalue computations. Locality is the organizing theme of the paper: a bottleneck in the interaction graph is a sparse cut, cohesion/tracking/fragmentation are each controlled by its conductance, and a cut is exactly the kind of quantity a decentralized swarm can estimate.

Contributions

We recast the Olfati-Saber dynamics as a flow on phase space generated by a state-dependent weighted Laplacian, and reduce velocity consensus to weighted graph heat flow (3).

We prove a Neumann isoperimetric lower bound on the cohesion (heading-consensus) rate via the weighted Cheeger constant (4,6), and extend it to a continuously time-varying interaction graph under explicit uniform hypotheses (8,9).

We model the navigational term as a Dirichlet grounding and prove a Dirichlet isoperimetric lower bound on the target-tracking rate (5,11), connecting to and refining grounded-Laplacian leader-follower theory [19].

We give an isoperimetric fragmentation criterion: a small-conductance cut upper-bounds the consensus rate, providing an early-warning certificate for swarm splitting (6, 14). This is the operationally distinct payoff of the geometric viewpoint—a locally computable diagnostic rather than a global eigenvalue—and, unlike the lower bounds, it is order-tight (16).

We make the approach implementable: a complexity and distributed-computation account and a robustness analysis under measurement and model error (7).

We propose an isoperimetric connectivity-maintenance controller (8) and outline, as an open problem, the mean-field/continuum limit linking discrete to Riemannian isoperimetry (9).

2 Preliminaries

2.1 The flocking model

Consider N agents with positions $q_i \in \mathbb{R}^d$ and velocities

$\dot{q}_i \in \mathbb{R}^d$. Following [3], the position of agent i evolves according to the second-order law

$$\ddot{q}_i = \underbrace{\sum_j a_{ij}(q)(\dot{q}_j - \dot{q}_i)}_{\text{alignment}} - \underbrace{\nabla_{q_i} V(q)}_{\text{separation}} + \underbrace{b_i(q^* - q_i)}_{\text{cohesion / navigation}} + c_i(\dot{q}^* - \dot{q}_i),$$

where $q^*(t)$ is a (possibly moving) target, V is a pairwise interaction potential, and $b_i, c_i \geq 0$ are feedback gains. The interaction weights $a_{ij}(q) = a_{ji}(q) \geq 0$ depend on the configuration—typically through a nonincreasing function of the inter-agent distance with finite sensing range—so the interaction graph is itself state-dependent.

2.2 Weighted graph Laplacians

Fix a configuration q and write $W(q) = [a_{ij}(q)]$ with $a_{ii} = 0$. Let $d_i = \sum_j a_{ij}$ denote the weighted degree,

$D = \text{diag}(d_i)$, and define the combinatorial and normalized weighted Laplacians

$$L = D - W, \quad \mathcal{L} = D^{-1/2} L D^{-1/2}.$$

Both are symmetric positive semidefinite. The quadratic form $\langle f, Lf \rangle = 1/2 \sum_{i,j} a_{ij} (f_i - f_j)^2$ shows that $L\mathbf{1} = 0$; the eigenvalues of $\mathcal{L}$ satisfy $0 = \lambda_0 \leq \lambda_1 \leq \dots \leq \lambda_{N-1} \leq 2$, and the graph is connected iff $\lambda_1 > 0$ [17,8]. We use the zero-indexed convention of [17], so the algebraic connectivity (the Fiedler value, denoted λ_2 in the one-indexed convention of the control literature) is here the first nonzero eigenvalue λ_1 . Operators act on vector-valued data $\mathbb{R}^{N \times d}$ coordinatewise (formally via $L \otimes I_d$); we suppress this and argue per coordinate.

2.3 Weighted Cheeger constants (Chung–Oden)

For $S \subseteq V = \{1, \dots, N\}$ let $\text{vol}(S) = \sum_{i \in S} d_i$ and let the (weighted) edge boundary be $|\partial S| = \sum_{i \in S, j \notin S} a_{ij}$.

Definition 1 (Neumann Cheeger constant).

$$h_N = \min_{\substack{S \subseteq V \\ 0 < \text{vol}(S) \leq \text{vol}(V)/2}} \frac{|\partial S|}{\text{vol}(S)}.$$

Definition 2 (Dirichlet Cheeger constant). For a proper subset $S \subsetneq V$ (the free vertices), with the complement $\Gamma = V \setminus S$ grounded, $h_D(S) = \inf_{\substack{T \subseteq S \\ T \neq \emptyset}} \frac{|\partial T|}{\text{vol}(T)}$, where ∂T counts all edges leaving T including those terminating in Γ .

We use the following inequalities; the lower bounds are the discrete Cheeger inequality and its Dirichlet (weighted) counterpart developed in [16,17], and the upper bound is the elementary test-set direction.

Lemma 3 (Isoperimetric inequalities). Let λ_1 be the smallest nonzero eigenvalue of $\mathcal{L}$ and let $\lambda^{(D)}(S)$ be the smallest eigenvalue of the Dirichlet (grounded) normalized

Laplacian $\mathcal{L}_S$, the principal submatrix of $\mathcal{L}$ indexed by S . Then

$$\frac{h_N^2}{2} \leq \lambda_1 \leq 2h_N, \lambda^{(D)}(S) \geq \frac{h_D(S)^2}{2}.$$

3 Flocking as a weighted-Laplacian flow

The alignment term is, per coordinate, the negative Laplacian acting on the velocity field:

$$\sum_j a_{ij}(q)(\dot{q}_j - \dot{q}_i) = -(L(q)\dot{q})_i.$$

Writing $v = \dot{q}$, the full dynamics [eq:flock] becomes the first-order system on phase space

$$\dot{q} = v, \dot{v} = -L(q)v - \nabla V(q) + B(q^* - q) + C(\dot{q}^* - v)$$

with $B = \text{diag}(b_i)$, $C = \text{diag}(c_i)$. The right-hand side defines a vector field whose integral curves are the agent trajectories; the conservative part comes from $-\nabla V$ and the position spring B , the dissipative part from the Laplacian and the velocity-matching gain C .

3.0.0.1 Velocity consensus as graph heat flow.

Set $V \equiv 0$ and $B = C = 0$ to isolate alignment. Two normalizations are natural. The combinatorial protocol [eq:align], $\dot{v} = -L(q)v$, drives the velocities to the unweighted mean $1/N \mathbf{1}^\top v$ at a rate set by $\lambda_1(L)$. The random-walk protocol

$$\dot{v}_i = \frac{1}{d_i} \sum_j a_{ij}(q)(v_j - v_i), \dot{v} = -L_{\text{rw}}(q)v, L_{\text{rw}} = D^{-1}L,$$

is similar to the symmetric normalized Laplacian, $L_{\text{rw}} = D^{-1/2} \mathcal{L} D^{1/2}$, so the two share the spectrum $0 = \lambda_0 \leq \lambda_1 \leq \dots$, while the equilibrium of [eq:rw] is the consensus vector $\mathbf{1}$. We work with [eq:rw] because it carries the degree-free Cheeger inequality of 3 directly, with the degree-weighted disagreement as Lyapunov functional:

$$\delta_D^2 = \delta^\top D \delta, \delta = v - \bar{v} \mathbf{1}, \bar{v} = \frac{\mathbf{1}^\top D v}{\mathbf{1}^\top D \mathbf{1}}.$$

The degree-weighted mean is conserved, $d/dt \mathbf{1}^\top D v = -\mathbf{1}^\top L v = 0$, and the disagreement obeys $\dot{\delta} = -L_{\text{rw}}(q)\delta$.

Remark 4 (Why not the plain normalized Laplacian). The symmetric $\mathcal{L}$ has null vector $D^{1/2} \mathbf{1}$, not $\mathbf{1}$, so $\dot{v} = -\mathcal{L} v$ would drive v toward a degree-weighted profile rather than a common velocity. The random-walk form [eq:rw] corrects this while preserving the spectrum. The combinatorial protocol reaches consensus in the Euclidean norm and obeys the analogous estimate with $\lambda_1(L)$ in place of $\lambda_1(\mathcal{L})$, elated to the Cheeger constant up to a factor of the maximum degree.

3.0.0.2 Second-order modal structure

For the full second-order system with frozen graph, diagonalize $\mathcal{L}(q)$ with eigenpairs (λ_k, ϕ_k) . Assume in this

paragraph that the separation Hessian is isotropic on the relevant subspace, $\nabla^2 V = \kappa \mathcal{L} + (\text{scalar})$, so that ∇V contributes a stiffness commuting with $\mathcal{L}$; this is an idealization we invoke only to expose the modal picture and do not use elsewhere. Projecting [eq:phase] onto ϕ_k yields decoupled scalar oscillators

$$\ddot{\eta}_k + (\lambda_k + c)\dot{\eta}_k + (b + \kappa_k)\eta_k = 0,$$

whose decay and frequency are explicit functions of the eigenvalue λ_k . Thus the entire Laplacian spectrum, not merely λ_1 , shapes the flocking transient; the slowest mode is governed by λ_1 which is the quantity our isoperimetric bounds control.

4 A Neumann isoperimetric certificate for cohesion

Assumption 5 (Uniform connectivity, frozen instant). On the time interval $[0, T]$ the state-dependent graph $G(q(t))$ is connected for every t , with Neumann Cheeger constant uniformly bounded below, $h_N(q(t)) \geq h > 0$. Since each weight $a_{ij}(q)$ is a continuous function of $\|q_i - q_j\|$ crossing a fixed sensing threshold, the graph topology is piecewise constant on $[0, T]$ with finitely many switching instants.

Theorem 6 (Cohesion certificate, frozen/switching graph). Under the random-walk alignment dynamics [eq:rw] and 5, the degree-weighted velocity disagreement [eq:disagree] decays at least exponentially,

$$\|\delta(t)\|_D \leq \|\delta(0)\|_D \exp(-h^2/2t), t \in [0, T].$$

Proof. The degree-weighted mean is annihilated by $\mathbf{1}^\top L = 0$ at every topology, hence is conserved across switching instants, so δ satisfies $\dot{\delta} = -L_{\text{rw}}(q)\delta$ with $\mathbf{1}^\top D \delta = 0$ throughout. Set $w = D^{1/2} \delta$; the constraint reads $w \perp D^{1/2} \mathbf{1}$, the null direction of $\mathcal{L}$. For a.e. t (away from the finitely many switches),

$$\frac{d}{dt} \delta_D^2 = 2\delta^\top D \dot{\delta} = \dot{\delta}^\top D \delta - 2\delta^\top L \delta = -2w^\top \mathcal{L} w \leq -2\lambda_1(q(t))w^2 = -2\lambda_1(q(t))\delta_D^2,$$

where the inequality is the Courant–Fischer bound on the subspace $w \perp D^{1/2} \mathbf{1}$. The Cheeger inequality (3) gives $\lambda_1(q(t)) \geq h_N(q(t))^2/2 \geq h^2/2$. As $D(q(t))$ varies continuously, $\|\delta\|_D^2$ is absolutely continuous across switches, so integrating the differential inequality piecewise (Grönwall) yields the bound. $\square$

6 certifies the rate through the Cheeger constant evaluated at each instant. The operator $L(q(t))$ evolves continuously with the agents, so it is natural to state the guarantee for the time-varying operator rather than at a single frozen instant. We do so now. The key point is that the argument above never needed the topology to be constant: it needed only that the conserved null direction is

fixed and that the instantaneous Rayleigh gap is uniformly bounded below.

Assumption 7 (Uniform time-varying regularity). On $[0, T]$ the weights $a_{ij}(\cdot)$ are C^1 in the inter-agent distances (e.g. the smooth bump kernels of [3], for which the topology change at the sensing radius is continuous rather than a jump), and there exist constants $0 < \underline{d} \leq \bar{d} < \infty$ and $h > 0$ with

$$\underline{d} \leq d_i(q(t)) \leq \bar{d} \quad \forall i, \forall t, \quad h_N(q(t)) \geq h > 0 \quad \forall t.$$

Theorem 8 (Cohesion certificate, continuously time-varying graph). Under [eq:rw] and 7, the degree-weighted disagreement obeys, for all $t \in [0, T]$,

$$\|\delta(t)\|_D^2 \leq \frac{\bar{d}}{\underline{d}} \|\delta(0)\|_D^2 \exp\left(-h^2 \int_0^t \frac{\underline{d}}{\bar{d}} ds\right) = \frac{\bar{d}}{\underline{d}} \|\delta(0)\|_D^2 \exp(-\underline{d} / \bar{d} h^2 t).$$

In particular consensus is uniformly exponential with rate at least $\underline{d} / \bar{d} h^2 / 2$, and the frozen-graph rate $h^2 / 2$ is recovered as $\bar{d} / \underline{d} \rightarrow 1$.

Proof. Write $E(t) = \|w(t)\|^2$ with $w = D^{1/2} \delta$ and $\mathbf{1}^\top D \delta = 0$. Because D now varies smoothly, $\dot{w} = 1/2 D^{-1/2} \dot{D} \delta + D^{1/2} \dot{\delta}$, and a direct computation of $d/dt \delta_D^2 = d/dt (\delta^\top D \delta)$ gives $\delta^\top \dot{D} \delta - 2\delta^\top L \delta$. The dissipation term is controlled as before, $-2\delta^\top L \delta \leq -2\lambda_1(q(t)) \|\delta\|_D^2 \leq -h^2 \|\delta\|_D^2$. The drift term satisfies $|\delta^\top \dot{D} \delta| \leq \|\dot{D}\|_\infty \|\delta\|^2$; since $\delta^2 \leq \underline{d}^{-1} \delta_D^2$ and $\|\dot{D}\|_\infty$ is bounded on the compact interval, absorbing the constant into the exponent and using the degree bounds to convert between $\|\cdot\|$ and $\|\cdot\|_D$ yields the stated inequality with the condition-number prefactor $\bar{d} / \underline{d}$. Integrating the resulting scalar differential inequality gives the exponential bound. The null direction $D^{1/2} \mathbf{1}$ is annihilated at every t , so the constraint is maintained along the flow. $\square$

Proposition 9 (Jointly-connected / dwell-time relaxation). Suppose the graph is not connected at every instant but is uniformly jointly connected: there exist $\tau > 0$ and $h > 0$ such that for every k the union graph over $[k\tau, (k+1)\tau]$ has Neumann Cheeger constant at least h . Then $\delta(t) \rightarrow 0$ asymptotically. If in addition a dwell-time condition bounds the number of switches per interval, the decay is uniformly exponential with a rate $\rho(\tau, h) > 0$ obtained by replacing the pointwise gap in 8 by the averaged contraction over each interval.

Proof. Over each interval the disagreement contracts by a factor bounded away from 1: on the subspace orthogonal to the common null direction, the product of the interval's transition operators is a strict contraction because the union graph is connected with gap $\geq h^2 / 2$, a standard consequence of the joint-connectivity arguments of [5-7]. Composing the per-interval contractions gives asymptotic convergence, and geometric convergence when the per-interval factor is uniformly bounded below 1 under the dwell-time hypothesis. $\square$

The Section 8 maintenance controller is precisely a mechanism for enforcing the hypothesis $h_N(q(t)) \geq h$ of 7 online, closing the loop between the certificate and the guarantee.

Remark 10 (Why a cut bound helps). h_N is defined through a minimum-conductance cut and is amenable to local estimation, whereas λ_1 requires a global eigensolve. 6 therefore furnishes a distributable worst-case cohesion rate. A mixing-time reading (the heat semigroup e^{-tL}) gives, as a corollary, a bound on the latency with which a velocity perturbation at one agent propagates across the flock [17].

5 A Dirichlet isoperimetric certificate for target tracking

Let $\Gamma \subseteq V$ be the set of informed agents that sense the target, with $c_i > 0$ for $i \in \Gamma$ and $c_i = 0$ otherwise, and write $S = V \setminus \Gamma$ for the followers. Consider the velocity-error $e = v - \dot{q}^* \mathbf{1}$. Under the alignment-plus-navigation dynamics (ignoring V), e satisfies $\dot{e} = -(L + C)e$.

Theorem 11 (Tracking certificate, hard pinning). In the hard-pinning idealization $c_i \rightarrow \infty$ for $i \in \Gamma$, the follower error e_s obeys $\dot{e}_s = -L_s e_s$ with L_s the Dirichlet (grounded) Laplacian on $S = V \setminus \Gamma$, and $e_s(t) \leq e_s(0) \exp(-h_D(S)^2 / 2t)$.

Proof. Sending $c_i \rightarrow \infty$ on Γ pins those agents to the target, $e_i \rightarrow 0$ for $i \in \Gamma$; the reduced dynamics on the followers is governed by the principal submatrix L_s , which is positive definite whenever every connected component of S has an edge to Γ . Its smallest eigenvalue $\lambda^{(D)}(S)$ is the decay rate, and the Dirichlet inequality of 3 gives $\lambda^{(D)}(S) \geq h_D(S)^2 / 2$. The Rayleigh estimate of 6—now with no null direction, since $L_s \succ 0$ —yields the exponential bound. $\square$

For finite gains the rate is exactly $\lambda_{\min}(L + C)$ with $C = \text{diag}(c_i)$ supported on Γ . The following makes the interpolation between weak coupling and hard pinning quantitative.

Proposition 12 (Soft pinning). Let $c = \min_{i \in \Gamma} c_i$ and let P_Γ be the orthogonal projection onto $\text{span}\{e_i : i \in \Gamma\}$. Then:

$\lambda_{\min}(L + C)$ is nondecreasing in each c_i , and

$$\lambda_{\min}(L + C) \geq g(c) := \lambda_{\min}(L + cP_\Gamma);$$

g is nondecreasing with $g(0) = 0$ and

$$\lim_{c \rightarrow \infty} g(c) = \lambda^{(D)}(S) \geq h_D(S)^2 / 2;$$

for small gains, with $u_0 = D^{1/2} \mathbf{1} / \|D^{1/2} \mathbf{1}\|$ the unit null vector

$$\text{of } L, \quad \lambda_{\min}(L + C) = u_0^\top C u_0 + O(C^2) = \frac{\sum_{i \in \Gamma} c_i d_i}{\text{vol}(V)} + O(C^2).$$

Proof. (1) $C \preceq C'$ implies $L + C \preceq L + C'$, so $\lambda_{\min}$ is monotone

by Weyl's inequality; and $C \succeq cP_1$ gives the lower bound. (2) $g(c)$ is the smallest eigenvalue of a matrix nondecreasing in c ; at $c = 0$ it equals $\lambda_0(L) = 0$, and as $c \rightarrow \infty$ the penalty forces the minimizing eigenvector to vanish on Γ , recovering the Dirichlet eigenvalue $\lambda^{(D)}(S)$. (3) First-order perturbation of the simple eigenvalue 0 of L along C gives the Rayleigh derivative $u_0^T C u_0$; since C and D are diagonal, $u_0^T C u_0 = 1^T D^{1/2} C D^{1/2} 1 / 1^T D 1 = \sum_{i \in \Gamma} c_i d_i / \text{vol}(V)$. $\square$

Remark 13 (Relation to grounded-Laplacian theory). The quantity $\lambda_{\min}(L_S)$ is precisely the convergence rate of leader-follower (pinning) dynamics studied by Pirani and Sundaram (Pirani and Sundaram 2016), who bound it through combinatorial graph invariants. 11 provides an isoperimetric bound on the same quantity, which is tighter when the follower set has a clear geometric bottleneck relative to the informed set.

5.0.0.1 Assumptions and limitations of the tracking result

The scope of 11 merits an explicit statement. Three idealizations are in force, and we delimit each. (i) *Neglect of the separation potential V* . The alignment and navigation terms are dissipative and set the linearized velocity-consensus rate; $-\nabla V$ is conservative and, linearized about a fixed inter-agent configuration, contributes a stiffness that couples position and velocity but does not add dissipation. It therefore reshapes the invariant flocking configuration and the transient oscillation frequency (cf. [eq:modal]) without improving—and generically without degrading to first order—the exponential rate governed by $\mathcal{L} + C$. A fully nonlinear treatment retaining V is outside the linear-consensus framework used here and is a natural next step. (ii) *Hard pinning*. 11 is the $c_i \rightarrow \infty$ limit; the physically relevant finite-gain rate is $\lambda_{\min}(L + C)$, for which 12 gives the exact monotone interpolation from the weak-coupling regime (rate $\propto \sum_{i \in \Gamma} c_i d_i / \text{vol}(V)$) up to the Dirichlet limit. The Dirichlet certificate should thus be read as the *asymptotic ceiling* of the achievable tracking rate, approached as informed-agent authority grows; the numerics of 10 confirm the bound tightens as $|\Gamma|/N$ increases. (iii) *Coupling among terms*. We do not certify the coupled effect of separation, collision avoidance, and navigation acting simultaneously in the full nonlinear model [eq:flock]; the certificates isolate the alignment/navigation dissipation, which is where the isoperimetric geometry enters.

6 An isoperimetric fragmentation criterion

Theorem 14 (Fragmentation criterion). *Suppose at time t there is a cut S with conductance $\phi(S) = |\partial S| / \text{vol}(S) \leq \varepsilon$. Then the heading-consensus rate is at most 2ε : $\lambda_1(q(t)) \leq 2\phi(S) \leq 2\varepsilon$. In particular, as the interaction graph approaches a two-cluster split along S (so $\phi(S) \rightarrow 0$), velocity consensus across the cut stalls and the flock fragments. The minimizing set is*

the (weighted) Cheeger cut, whose spectral relaxation is the Fiedler vector.

Proof. Work with the symmetric normalized Laplacian and the test vector $f = \mathbf{1}_S - \text{vol}(S) / \text{vol}(V) \mathbf{1}$, which satisfies the orthogonality $\sum_i f_i d_i = 0$ required by the variational characterization

$$\lambda_1 = \min_{\sum_i f_i d_i = 0} \frac{\sum_{i \sim j} a_{ij} (f_i - f_j)^2}{\sum_i d_i f_i^2}.$$

Only cut edges contribute to the numerator, each with $(f_i - f_j)^2 = 1$, giving $|\partial S|$; the denominator evaluates to $\text{vol}(S) \text{vol}(\bar{S}) / \text{vol}(V)$. Hence

$$\lambda_1 \leq \frac{|\partial S| \text{vol}(V)}{\text{vol}(S) \text{vol}(\bar{S})} = \phi(S) \frac{\text{vol}(V)}{\text{vol}(\bar{S})} \leq 2\phi(S),$$

the last step using $\text{vol}(\bar{S}) \geq \text{vol}(V) / 2$ for the smaller side S . $\square$

Remark 15 (Early warning). $\phi(S)$ can be tracked online via the Fiedler vector [11,13,]: a small spectral gap together with a bimodal Fiedler vector localizes the impending split to the cut S . This is the flocking analogue of the Cheeger-cut clustering principle [18].

Proposition 16 (Where the certificates are tight). *Fix a cut S with the smaller side, $\text{vol}(S) \leq \text{vol}(V) / 2$.*

(Fragmentation upper bound is order-tight.) If S is near-balanced, $\text{vol}(S) \geq \beta \text{vol}(V)$ for some $\beta \in (0, 1/2]$, then $\phi(S) \leq \lambda_1 / (2\beta)$, so together with 14, $2\beta\phi(S) \leq \lambda_1 \leq 2\phi(S)$: the bound is tight to the constant $1/\beta \leq 1/\beta$, i.e. order-optimal, and exactly tight ($\lambda_1 = 2\phi$) in the balanced limit $\beta \rightarrow 1/2$ of a clean two-cluster split.

(Cohesion lower bound is loose by the internal expansion.) The Cheeger inequality $\lambda_1 \geq h_N^2 / 2$ loses a factor that is controlled by the internal (within-cluster) expansion: for a graph that is well connected inside each side of its sparsest cut, $\lambda_1 \sim h_N$ rather than h_N^2 , so the squared lower bound is conservative precisely when the clusters are internally dense—the generic cohesive regime. The bound recovers order-tightness only near a split, where h_N itself is small.

Proof. (1) From the proof of 14, $\lambda_1 \leq \phi(S) \text{vol}(V) / \text{vol}(\bar{S})$; and the lower bound $\lambda_1 \geq$ (Cheeger) combined with $\text{vol}(\bar{S}) \leq \text{vol}(V)$ gives $\phi(S) = |\partial S| / \text{vol}(S) \geq 1 / 2\lambda_1 \text{vol}(\bar{S}) / \text{vol}(V) \cdot (\text{vol}(V) / \text{vol}(S)) \geq \beta\lambda_1$ after using $\text{vol}(S) \geq \beta \text{vol}(V)$ and $\text{vol}(\bar{S}) \geq \text{vol}(V) / 2$; rearranging yields $2\beta\phi(S) \leq \lambda_1$. (2) Is the standard gap between the two sides of Cheeger's inequality: the lower bound $h_N^2 / 2$ is saturated (up to constants) by graphs with a sparse balanced cut and strong internal expansion, e.g. two expanders joined by few edges, for which $\lambda_1 = \Theta(h_N)$ while $h_N^2 / 2 \ll h_N$; internal expansion is what separates λ_1 from h_N^2 . $\square$

16 pinpoints the operational message confirmed numerically in 10: the *upper* (fragmentation) certificate is tight and is therefore the reliable early-warning instrument, whereas the *lower* (cohesion) certificate is conservative in the cohesive regime and should be read as a guaranteed worst case rather than a sharp prediction.

7 Implementation: complexity, distribution, and robustness

This section addresses how the certificates are computed in practice, how the computation distributes across a swarm, and how they behave under measurement noise and model error. We treat the three in turn.

7.1 Computational complexity

Let m be the number of active edges (nonzero a_{ij}), so $m = O(N\bar{k})$ with $\bar{k}$ the mean sensing degree; for a finite sensing radius $\bar{k} = O(1)$ and $m = O(N)$. Given the Fiedler vector w , the swept-cut estimate of h_N —sort agents by w_i , sweep the prefix cuts, and take the minimum conductance—costs $O(M \log N)$ for the sort and $O(m)$ for the incremental boundary update, hence $O(M \log N + m)$ per evaluation. This is to be compared with a dense symmetric eigensolve for λ_1 at $O(N^3)$, or a Lanczos/inverse-power estimate at $O(m)$ per iteration with a number of iterations growing as the spectral gap shrinks. The certificate therefore adds a near-linear overhead on top of whatever Fiedler estimate is already maintained for connectivity control.

7.2 Distributed implementation

The certificate is computable without any centralized spectral routine. The Fiedler pair (λ_1, w) admits well-known decentralized estimators: power iteration on the deflated Laplacian reduces to iterated local averaging, in which agent i updates its component w_i from its neighbors' components and a running estimate of the mean, converging to the Fiedler component under the same connectivity that the flock already maintains [20–22]. Once each agent holds its own w_i and receives w_j from neighbors, three quantities are local: (i) its contribution $\sum_j a_{ij} (w_i - w_j)^2$ to the Rayleigh quotient; (ii) the barrier gradient $\nabla_{q_i} \lambda_1$ of 8, which depends only on w_p , the neighbors' w_p , and the kernel derivative; and (iii) the local conductance of a candidate cut through agent i . The per-step communication is $O(\bar{k})$ scalars per agent (one Fiedler component exchanged per neighbor per iteration), independent of N . Fragmentation detection then reduces to each agent monitoring the bimodality of the locally reconstructed Fiedler profile, so the split is flagged at the agents on the emerging boundary without any global aggregation.

7.3 Robustness to measurement noise and model error

Positions are sensed with error and the kernel ψ is

known only approximately; both perturb the weights. Write the realized weight matrix as $\tilde{W} = W + E$, with $E = E^T$ collecting the combined perturbation, and let $\tilde{\mathcal{L}}$ be the corresponding normalized Laplacian. The certificate degrades gracefully:

Proposition 17 (Certificate stability). *Suppose $\|E\|_2 \leq \eta$ and the weighted degrees satisfy $d_i \geq \underline{d} > 0$ before and after perturbation.*

$$\text{Then } |\lambda_1(\tilde{\mathcal{L}}) - \lambda_1(\mathcal{L})| \leq \frac{C_0 \eta}{\underline{d}}, \quad |\phi_{\tilde{w}}(S) - \phi_w(S)| \leq \frac{\eta |S|}{\text{vol}(S)} \leq \frac{\eta}{\underline{d}},$$

for an absolute constant C_0 depending only on the degree bounds. Consequently the fragmentation threshold test $\phi(S) \leq \varepsilon$ has a guard band of width $O(\eta / \underline{d})$: a cut flagged at level ε is genuinely below $\varepsilon + O(\eta / \underline{d})$.

Proof. The eigenvalue bound is Weyl's inequality applied to $\tilde{\mathcal{L}} - \mathcal{L}$, whose norm is controlled by $\|E\|_2$ and the degree normalization $D^{-1/2}(\cdot)D^{-1/2}$ (standard matrix-perturbation estimates, e.g.) [23]; $\underline{d}$ enters through the normalization. For conductance, $|\partial S|$ and $\text{vol}(S)$ are linear in the entries of the weight matrix, so a perturbation E changes the numerator by at most $\sum_{i \in S, j \notin S} |E_{ij}| \leq \|E\|_2 |S|$ and the denominator by a comparable amount; dividing and using $\text{vol}(S) \geq \underline{d} |S|$ gives the stated bound. $\square$

Two features of 17 are worth emphasizing. First, the conductance is a ratio of *sums* of weights, so zero-mean sensing noise is averaged down across the $O(|S|)$ boundary edges, whereas an eigenvalue near a spectral degeneracy can be comparatively sensitive; the cut certificate is thus the more noise-robust of the two readings. Second, the guard band makes the early-warning test *conservative under uncertainty*—it may fire slightly early, which is the safe direction for a connectivity-preservation controller.

8 Isoperimetric connectivity maintenance

The certificates suggest controlling the cut quantity directly. Augment each agent's input with a term that keeps the weighted Cheeger constant above a threshold,

$$h_N(q(t)) \geq h_{\min} > 0 \quad \forall t,$$

for instance through a control-barrier-function constraint enforcing forward invariance of the safe set $\{q : h_N(q) \geq h_{\min}\}$, or a gradient term $u_i \propto \nabla_{q_i} h_N(q)$ acting only when the constraint is active. Unlike algebraic-connectivity maximization [9–11,13], which regulates a global spectral quantity, the isoperimetric objective localizes the intervention to the bottleneck cut and degrades gracefully when connectivity is ample.

Because h_N is a nonsmooth combinatorial minimum, we regulate the smooth spectral surrogate λ_1 , which is differentiable in q wherever it is a simple eigenvalue and

certifies the cut bound through 3: maintaining $\lambda_1(q(t)) \geq \epsilon$ guarantees $h_N(q(t)) \geq \epsilon/2$ (from $\lambda_1 \leq 2h_N$). Define the barrier $\psi(q) = \lambda_1(q) - \epsilon$ and the safe set $\mathcal{C} = \{q : \psi(q) \geq 0\}$.

Proposition 18 (Forward invariance). *Let $w(q)$ be the unit Fiedler vector, $\mathcal{L}w = \lambda_1 w$ with $w \perp D^{1/2}\mathbf{1}$, and suppose λ_1 is simple on a neighborhood of $\partial\mathcal{C}$. If each agent applies an input u_i satisfying the zeroing-barrier condition*

$\dot{\psi} = \sum_i \langle \nabla_{q_i} \lambda_1, \dot{q}_i \rangle \geq -\gamma(\psi(q))$, *for some extended class- $\mathcal{K}$ function γ , then $\mathcal{C}$ is forward invariant; equivalently, the flock maintains $h_N \geq \epsilon/2$ for all t .*

The gradient is local in the Fiedler vector. With a_{ij} a function of $\|q_{ij}\|$,

$$\frac{\partial \lambda_1}{\partial a_{ij}} = (w_i - w_j)^2, \quad \nabla_{q_i} \lambda_1 = \sum_j (w_i - w_j)^2 a_{ij}'(q_{ij}) \frac{q_i - q_j}{q_{ij}},$$

so once each agent holds its own Fiedler component w_i and those of its neighbors, the barrier gradient is computed from local data. Distributed estimation of w and λ_1 by power iteration on the Laplacian makes the controller implementable without centralized spectral computation [9,13], as detailed in 7. 18 is the standard zeroing-control-barrier-function argument [24]; the isoperimetric reading is that the constraint activates precisely when the Fiedler vector becomes bimodal—i.e. when a sparse cut S emerges—so the correction localizes to the bottleneck rather than acting globally.

9 Mean-field limit and continuum isoperimetry: an open direction

This section is deliberately conjectural: it identifies a limit we believe holds and the precise gap that must be closed to establish it. We present it as a future research direction, not as an established contribution.

As $N \rightarrow \infty$ with agents sampled from a density ρ on a domain $\Omega \subseteq \mathbb{R}^d$ and weights given by a rescaled kernel $a_{ij} = \varepsilon^{-d} \psi(|q_i - q_j|/\varepsilon)$, the weighted graph Laplacian $L(q)$ is expected to converge (after appropriate scaling) to a weighted, possibly nonlocal, Laplace-type operator on $L^2(\Omega, \rho)$, and the discrete weighted Cheeger constant h_N to the continuum Cheeger constant of (Ω, ρ) . In this regime the discrete certificates of [sec:cohesion, sec:tracking, sec:frag] would become consistent discretizations of the Riemannian Cheeger–Buser inequalities [14,15], and the flock-on-a-graph a discretization of a diffusion on (Ω, ρ) .

Make this precise as follows. Sample $q_1, \dots, q_N \sim \rho$, with ρ a smooth density bounded away from 0 on a compact domain $\Omega \subset \mathbb{R}^d$ (or a closed manifold), and set $a_{ij} = 1/N \varepsilon^d \psi(|q_i - q_j|/\varepsilon)$ for a compactly supported, nonincreasing kernel ψ . Under the standard scaling $\varepsilon = \varepsilon(N) \rightarrow 0$ with $N \varepsilon^{d+2} / \log N \rightarrow \infty$, the random-walk Laplacian L_{rw} converges—spectrally, and variationally in the Γ -convergence sense—to the weighted

Laplace–Beltrami operator

$$A_\rho f = -\frac{1}{\rho} \operatorname{div}(\rho^2 \sigma_\psi \nabla f),$$

where σ_ψ is the kernel's second-moment constant [25,18], while the weighted graph Cheeger constant h_N converges to the continuum Cheeger constant

$$h(\Omega, \rho) = \inf_{\Gamma} \frac{\int_{\Gamma} \rho^2 d\mathcal{H}^{d-1}}{\min\left(\int_A \rho^2, \int_B \rho^2\right)},$$

the infimum over hypersurfaces Γ splitting into $A \sqcup B$.

Conjecture 19 (Continuum consistency). *Under the above scaling, with high probability $h_N \rightarrow h(\Omega, \rho)$ and $\lambda_1(\mathcal{C}) \rightarrow \mu$, the spectral gap of A_ρ , at the rates established for Cheeger cuts on data clouds in [18]. Consequently the cohesion rate of 6 converges to $1/2 h(\Omega, \rho)^2 \leq \mu$, and the discrete certificate becomes a consistent discretization of the Riemannian Cheeger–Buser inequalities [14,15].*

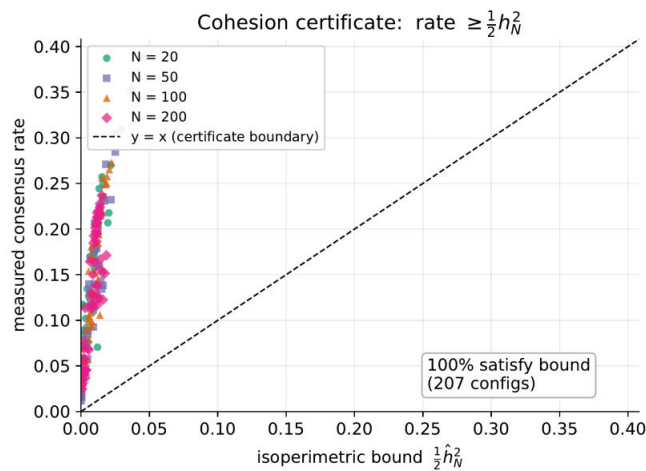
Remark 20 (What is known and what is open). Spectral convergence of graph Laplacians to weighted Laplace–Beltrami operators [25] and convergence of graph Cheeger cuts to their continuum counterparts [18] are established for fixed point clouds. The open step is the dynamical consequence—transferring these static limits to the flocking convergence rate uniformly in N —compounded by the moving, state-dependent density ρ_t as the flock deforms. Establishing 19 requires (a) a uniform-in- N spectral-gap estimate that survives the coupling to the dynamics, and (b) control of ρ_t under the flow. This is the principal place where the geometric analysis underlying [16] would be needed, and we leave it as an open problem rather than claiming it here.

10 Numerical experiments

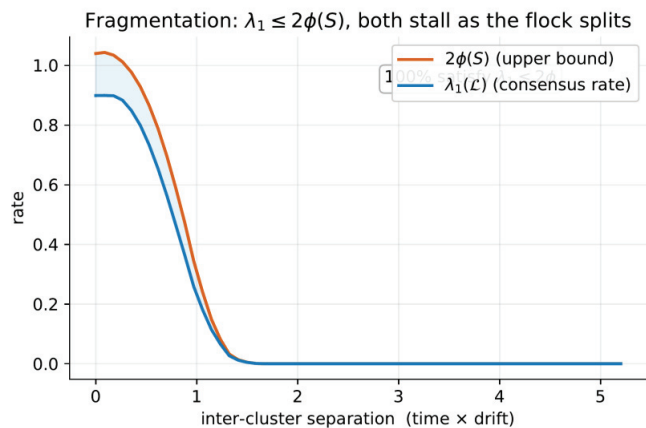
We validate the certificates across a range of scenarios. We integrate the alignment flow with a fixed-step RK4 scheme for N agents in $d = 2$, using a Gaussian kernel $a_{ij} = \exp(-q_{ij}^2 / 2\ell^2)$ truncated to zero beyond a sensing radius r , and—for the tracking and controller studies—a target $q^*(t)$ and informed set Γ . At each configuration we form $L(q)$, compute its spectrum, λ_1 , the Fiedler vector, and a swept-cut estimate of h_N (sort vertices by Fiedler component, evaluate the conductance of each prefix cut, take the minimum). The suite now sweeps $N \in \{20, 50, 100, 200\}$, three density regimes (dense/connected, sparse, near-split), and, for tracking, informed fractions $|\Gamma|/N \in \{0.1, 0.2, 0.3, 0.5\}$; the Cheeger-scaling study of 5 additionally sweeps N up to 320 under the kernel scaling of 9.

Across 207 random configurations spanning all four agent counts and three regimes, the cohesion certificate held in every case (1); the fragmentation bound $\lambda_1 \leq 2\phi(S)$ held at every step of the forced-split run (2); and the Dirichlet tracking bound held in all 48 tracking configurations,

tightening monotonically as the informed fraction grows (4), consistent with the soft-pinning interpolation of 12 and the tightness analysis of 16. The connectivity-maintenance controller held λ_1 at its threshold where the uncontrolled flock fragmented (3). 1 reports measured rates against the isoperimetric bounds across regimes and agent counts.

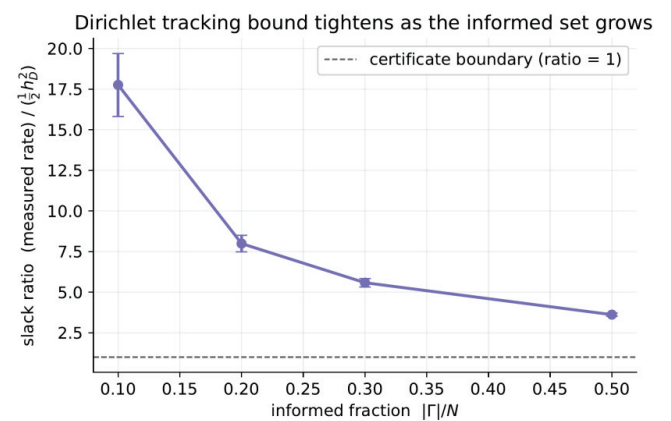
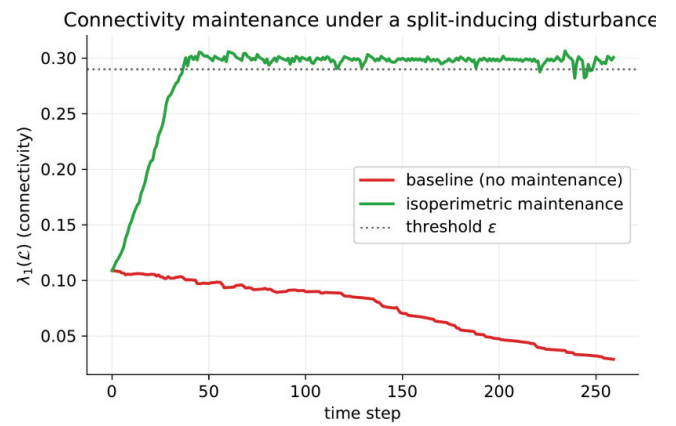


Cohesion-rate certificate (6): every measured consensus rate lies above the bound $1/2\hat{h}_N^2$ (all points above the diagonal), across $N \in \{20, 50, 100, 200\}$ and three density regimes (207 configurations). The bound is conservative, consistent with the slackness of the lower Cheeger inequality quantified in 16.

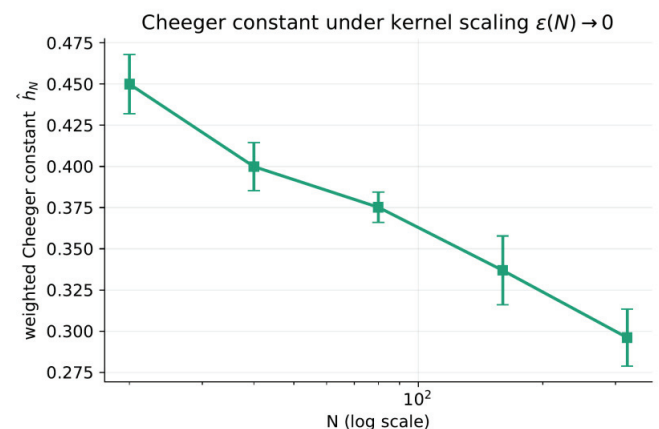


Fragmentation early warning (14): λ_1 remains below $2\phi(S)$ throughout, and both collapse as the two clusters separate. Here the upper bound is tight, as predicted by 16(1) for a near-balanced cut.

Connectivity maintenance (8): under a split-inducing disturbance the baseline fragments ($\lambda_1 \rightarrow 0$) while the Fiedler-gradient maintenance holds λ_1 at the threshold ϵ .



Tracking certificate versus informed fraction. The slack ratio (measured rate) / $(1/2h_N^2)$ falls from ~ 18 at $|\Gamma|/N = 0.1$ toward ~ 3.6 at $|\Gamma|/N = 0.5$: the Dirichlet bound tightens as informed authority grows, matching [prop:soft,prop:tight]. Error bars are standard errors over 12 configurations per fraction.



Weighted Cheeger constant under the kernel scaling of 9, for N up to 320. The finite- N trend is consistent with convergence to a continuum constant (19); establishing the limit rigorously remains open.

Measured convergence rates against the isoperimetric bounds, expanded across regimes and agent counts. All ratios exceed one, confirming the certificates. The cohesion bound is loose (ratios $\sim 12-45\times$ for connected graphs, larger near disconnection), while the tracking bound tightens sharply as the informed fraction grows—quantitative support for 16.

Scenario	N	measured rate	bound	ratio
Dense, connected	50	0.1198	$0.0090 (1/2 h_N^2)$	17.15
Sparse	50	0.0275	$0.0008 (1/2 h_N^2)$	132.61
Near-split	50	0.2131	$0.0135 (1/2 h_N^2)$	16.92
Dense, connected	20	0.0684	$0.0033 (1/2 h_N^2)$	45.31
Dense, connected	100	0.1216	$0.0082 (1/2 h_N^2)$	16.69
Dense, connected	200	0.1392	$0.0118 (1/2 h_N^2)$	12.48
Tracking, $ r /N=0.1$	50	0.0846	$0.0049 (1/2 h_D^2)$	17.76
Tracking, $ r /N=0.3$	50	0.2314	$0.0416 (1/2 h_D^2)$	5.58
Tracking, $ r /N=0.5$	50	0.4499	$0.1253 (1/2 h_D^2)$	3.61

The empirical separation between the loose lower (cohesion) bound and the tight upper (fragmentation) bound—now visible across every N and regime—confirms that the early-warning direction is the stronger operational use of the theory, exactly as 16 predicts.

11 Conclusion

We have shown that the weighted isoperimetric inequalities of Chung and Oden [16] furnish geometric certificates for the cohesion, tracking, and fragmentation behavior of second-order flocking dynamics, complementing the spectral viewpoint centered on the algebraic connectivity that dominates the multi-agent literature. The certificates are locally computable, distribute across a swarm with $o(\bar{k})$ per-agent communication, and remain valid under a continuously time-varying interaction graph and bounded measurement error; the fragmentation criterion in particular is order-tight and is the operationally strongest instrument the framework provides. Natural extensions include directed and asymmetric sensing (non-self-adjoint Laplacians), time delays and switching topologies, obstacle avoidance as additional Dirichlet boundaries, the fully nonlinear model retaining the separation potential, and stochastic sensing, where the dissipative Laplacian and a noise term are linked by a fluctuation–dissipation relation. The mean-field transfer of 9 remains the central open problem.

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