

Commanded Fission and Fusion of Flocks: Weighted Cheeger Certificates for Organized Disaggregation, Tracking, and Re-Aggregation

Kevin D Oden* and Maia Berkane

Kevin D Oden & Associates, LLC, 505 Montgomery Street, San Francisco, CA 94111, USA

ABSTRACT

Many missions require a flock to split apart on command into prescribed sub-groups, hold those sub-groups as coherent flocks while each tracks its own objective, and then re-aggregate on command. Existing fission–fusion controllers are largely bio-inspired heuristics without convergence guarantees. We give a certificate-based framework in which each phase of the maneuver is governed by a weighted isoperimetric (Cheeger) inequality. The organizing observation is that *organized disaggregation is the fragmentation criterion of flocking used as an objective rather than avoided as a hazard*: the same Fiedler-gradient that a connectivity-maintenance controller ascends to keep a flock together is descended, block-wise, to pull it apart along a commanded cut. We prove (i) a two-way fission certificate bounding the global consensus rate by the cross-cut conductance while each sub-flock retains a Neumann-certified cohesion rate; (ii) a *multiway* fission certificate a higher-order Cheeger bound $\mu_{k-1} \geq 2 \max_j \phi(V_j)$ that certifies commanded k -way splitting and yields a spectral-gap detector for completion; (iii) a per-group Dirichlet tracking certificate for the hold phase; and (iv) a Neumann re-cohesion certificate for commanded fusion. The resulting three-mode switched controller is analyzed with a dwell-time argument, and we address the eigenvalue-crossing at the split instant by handing off from the spectral surrogate to the conductance gradient, which stays single-valued. A simulated disaggregate–hold–re-aggregate cycle confirms every certificate: the global algebraic connectivity collapses to zero under fission while each block stays above threshold, the higher-order spectral gap opens, and connectivity is restored on the fusion command.

*Corresponding author

Kevin D Oden, Kevin D Oden & Associates, LLC, 505 Montgomery Street, San Francisco, CA 94111, USA.

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Introduction

Collective motion has been engineered from local rules since Reynolds and Vicsek et al., and given a control-theoretic foundation by Olfati-Saber and the consensus literature [1-7]. That foundation is overwhelmingly concerned with keeping a flock *together*: the algebraic connectivity (Fiedler value) of the interaction Laplacian sets the slowest disagreement mode, and connectivity-control design regulates it to preserve cohesion [8-12].

Many missions, however, require the opposite on command: a flock that *splits* into prescribed sub-groups, holds each as a coherent flock tracking its own objective, and then *re-aggregates*. We call this *fission–fusion*, keeping the term from behavioral ecology, where it names groups that repeatedly split and re-merge. The same phenomenon is central to two fields closer to the present setting. In the physics of active matter and collective motion it is the controlled counter-part of the *aggregation–fragmentation* of self-propelled ensembles indeed the companion paper’s *fragmentation* certificate is exactly the physics side of this vocabulary [13,14]. In engineering it is the *split–rejoin* maneuver of swarm robotics. The engineering treatments to date are mostly bio-inspired heuristics. A starling-inspired controller performs fission around a dynamic obstacle and re-fusion with a controllable sub-swarm size, monitored by a polarization index but without

spectral guarantees; a shepherding approach unifies separation of a single agent with connectivity maintenance of the remainder, but its authors note the performance is not yet theoretically guaranteed [15,16]. What is missing is a principled account in which the split is *commanded and certified*: split along a prescribed partition, with a guarantee that each piece remains a flock and that the whole can be reassembled.

We supply that account, and the tool is already latent in the flocking problem. The alignment term of the Olfati-Saber law is the action of a state-dependent weighted graph Laplacian, so the weighted isoperimetric inequalities of Chung and Oden apply [17]. In a companion paper we used this to certify cohesion (a Neumann Cheeger lower bound on the consensus rate), target tracking (a Dirichlet Cheeger lower bound), and *fragmentation*: a sparse cut S with conductance $\phi(S)$ upper-bounds the consensus rate, $\lambda_1 \leq 2\phi(S)$, so a flock splits as $\phi(S) \rightarrow 0$ [18]. There, fragmentation is a hazard to be detected and avoided. Here we run it forward: Organized disaggregation is the fragmentation criterion used as an objective. The Fiedler-gradient that a maintenance controller ascends to hold a flock together is descended, block-wise, to pull it apart along a commanded cut while each block’s own Cheeger constant is held above threshold so it remains a coherent flock.

Contributions

- We formulate commanded fission–fusion as a three-mode switched controller (split / hold / merge) built entirely from the isoperimetric gradient $\partial\lambda_1/\partial a_{ij} = (w_i - w_j)^2$, sign-flipped between severance and maintenance (Sections 3 and 7).
- We prove a **Two-way Fission Certificate** (Theorem 2): under commanded splitting, the global consensus rate is bounded by the cross-cut conductance and collapses, while each sub-flock retains a Neumann-certified cohesion rate; and we show the minimum-effort commanded cut is the Fiedler/Cheeger cut (Proposition 3).
- We prove a **Multway Fission Certificate** (Theorem 4): for a commanded k -partition Π , the $(k-1)$ -th normalized-Laplacian eigenvalue obeys $\mu_{k-1} \leq 2 \max_j \phi(V_j)$, a higher-order Cheeger bound that certifies k -way disaggregation and supplies a spectral-gap completion detector [19].
- We give a **Per-group Tracking Certificate** (Proposition 6) for the hold phase (block-wise Dirichlet grounding, connecting to containment control), a **Fusion Certificate** (Theorem 8) for commanded re-aggregation, a **Switched-Stability** result under dwell time (Theorem 10), and a treatment of the eigenvalue crossing at the split instant via handoff to the conductance gradient (Proposition 9)[20–22].
- We validate the whole cycle numerically (Section 8): every certificate holds through a com-manded $k=3$ disaggregate–hold–re-aggregate maneuver.

Preliminaries

Flocking as a Weighted Laplacian Flow Let N agents have positions $q_i \in \mathbb{R}^d$ and velocities $v_i = \dot{q}_i$. The Olfati-Saber alignment term is, per coordinate, $\sum_j a_{ij}(q)(v_j - v_i) = -(L(q)v)_i$, with symmetric state-dependent weights $a_{ij}(q) = a_{ji}(q) \geq 0$ (a nonincreasing function of $\|q_i - q_j\|$ with finite sensing range). With $W = [a_{ij}]$, degree $d_i = \sum_j a_{ij}$, $D = \text{diag}(d_i)$, the combinatorial and normalized weighted Laplacians are $L = D - W$ and $\mathcal{L} = D^{-1/2}LD^{-1/2}$, with eigenvalues $0 = \mu_0 \leq \mu_1 \leq \dots \leq \mu_{N-1} \leq 2$; the algebraic connectivity is $\lambda_1 := \mu_1$. The random-walk alignment flow $\dot{v} = -L_{\text{rw}}v$, $L_{\text{rw}} = D^{-1}L$, drives v to a common velocity, and the degree-weighted disagreement $\|\delta\|_D^2 = \delta^T D \delta$, $\delta = v - \bar{v}\mathbf{1}$, obeys $\dot{\delta} = -L_{\text{rw}}\delta$ [13].

Weighted Cheeger constants.

For $S \subseteq V = \{1, \dots, N\}$, $\text{vol}(S) = \sum_{i \in S} d_i$ and boundary $|\partial S| = \sum_{i \in S, j \notin S} a_{ij}$. The Neumann and Dirichlet Cheeger constants are

$$h_N = \min_{0 < \text{vol}(S) \leq \text{vol}(V)/2} \frac{|\partial S|}{\text{vol}(S)}, \quad h_D(S) = \inf_{T \subseteq S, T \neq \emptyset} \frac{|\partial T|}{\text{vol}(T)},$$

the latter with $V \setminus S$ grounded. For a partition $\Pi = \{V_1, \dots, V_k\}$ write the group conductance $\phi(V_j) = |\partial V_j| / \text{vol}(V_j)$ and the multiway expansion $\phi\Pi = \max_j \phi(V_j)$. We write $G[V_j]$ for the induced subgraph, with its own Neumann Cheeger constant $h_N(V_j)$ and algebraic connectivity $\lambda_1(V_j)$.

Base certificates from the companion paper. We use three results of [13], each a consequence of the weighted Cheeger inequalities [3, 4, 2]: with λ_1 the algebraic connectivity and $\lambda^{(D)}(S)$ the smallest grounded (Dirichlet) eigenvalue,

$$\frac{1}{2}h_N^2 \leq \lambda_1 \leq 2h_N, \quad \lambda^{(D)}(S) \geq \frac{1}{2}h_D(S)^2, \quad \lambda_1 \leq 2\phi(S) \quad \forall S. \quad (1)$$

The left inequality yields a cohesion rate $\geq \frac{1}{2}h_N^2$ (Neumann); the middle a tracking rate $\geq \frac{1}{2}h_D(S)^2$ (Dirichlet); the right is the fragmentation bound we now turn into a design objective.

Commanded Fission

Fix a commanded bipartition $\Pi = \{S, S^c\}$ and let $g(i) \in \{S, S^c\}$ denote agent i 's group. The split controller severs cross-group edges while defending each block's internal connectivity and steering each block to a commanded target.

Controller 1 (Two-way fission). For i in group $g = g(i)$, with target q_g^* and gains $\eta, \gamma, c > 0$,

$$u_i = -\eta \underbrace{\sum_{\ell: g(\ell) \neq g} \nabla_{q_i} a_{i\ell}}_{\text{sever cross-cut edges}} + \underbrace{\gamma \mathbb{1}[\lambda_1(V_g) < \epsilon] \nabla_{q_i} \lambda_1(V_g)}_{\text{defend block cohesion}} + \underbrace{c(q_g^* - q_i)}_{\text{navigate}} + u_i^{\text{col}}, \quad (2)$$

where u_i^{col} is a short-range collision term, and the block-connectivity gradient uses the induced subgraph Fiedler vector

$$w^g \text{ of } G[V_g] \text{ via } \nabla_{q_i} \lambda_1(V_g) = \sum_{j \in V_g} (w_i^g - w_j^g)^2 a'_{ij} (\|q_{ij}\|) \frac{q_i - q_j}{\|q_{ij}\|}.$$

The severance term is exactly the isoperimetric maintenance gradient of [13] with its sign reversed: maintenance ascends $\nabla \lambda_1$ to raise cross-cut weight, fission descends it to drive cross-cut weight to zero.

Theorem 2 (Two-way fission certificate). Suppose Controller 1 drives the cross-cut conductance $\phi(S)(t) \rightarrow 0$ while maintaining the induced Cheeger constants $h_N(S)(t), h_N(S^c)(t) \geq h > 0$. Then:

1. (Separation.) The global algebraic connectivity obeys $\lambda_1(t) \leq 2\phi(S)(t) \rightarrow 0$: cross-cut velocity consensus stalls and the flock disaggregates along S .
2. (Block cohesion.) On each side, the intra-block velocity disagreement contracts at rate at least $\frac{1}{2}h^2$; in the separated limit $\phi(S) = 0$ the two sides evolve as independent flocks, each certified by the Neumann bound (1).

Proof. (1) is the fragmentation inequality of (1) applied to the commanded cut S (test vector $f = \mathbf{1}_S - \frac{\text{vol}(S)}{\text{vol}(V)}\mathbf{1}$); it holds at every instant, so as the controller drives $\phi(S) \rightarrow 0$ the upper bound forces $\lambda_1 \rightarrow 0$. For (2), write the weight matrix in block form $W = \begin{pmatrix} W_S & C \\ C^T & W_{S^c} \end{pmatrix}$ with C the crossblock weights, $\|C\|_1 = |\partial S| \rightarrow 0$. The velocity flow restricted to S is $v_S = -L_{\text{rw}}(S)v_S + O(\|C\|)$; dropping the vanishing coupling gives the isolated block dynamics governed by $L(S)$, whose spectral gap satisfies $\lambda_1(S) \geq \frac{1}{2}h_N(S)^2 \geq \frac{1}{2}h^2$ by (1). The Rayleigh argument of [13] then contracts the intra-block disagreement at rate $\geq \frac{1}{2}h^2$; the $O(\|C\|)$ coupling during the transient is absorbed by the time-varying certificate (Sections 6 and 7).

Proposition 3 (Minimum-effort disaggregation). Define the instantaneous split effort as the crosscut weight mass that must be annihilated, $\mathcal{E}(S) = \sum_{i \in S, \ell \in S^c} a_{i\ell} = |\partial S|$. Among balanced bipartitions $\{S : \text{vol}(S) \in [\beta \text{vol}(V), (1-\beta) \text{vol}(V)]\}$, $\mathcal{E}(S)$ is minimized by the minimum-conductance (Cheeger) cut, whose spectral relaxation is the Fiedler vector. Hence commanded splitting along the Fiedler cut is the minimum-effort disaggregation.

Proof. $\mathcal{E}(S) = |\partial S|$ by definition. Under the volume/balance constraint, minimizing $|\partial S|$ is equivalent, up to the bounded volume normalization $\text{vol}(S) \in [\beta, 1-\beta] \text{vol}(V)$, to minimizing the conductance $\phi(S) = |\partial S| / \text{vol}(S)$; its minimizer is h_N , and the standard relaxation of the balanced minimum-conductance cut is thresholding the Fiedler vector [7, 4].

Multway Fission and the Higher-order Certificate

For a commanded k -partition the natural spectral object is not λ_1 but the $(k-1)$ -th eigenvalue. The following is the operational core of the paper: a genuinely k -way fission certificate, the easy (test-set)

direction of the higher-order Cheeger inequality [10], specialized to a commanded partition.

Theorem 4 (Multiway fission certificate). *Let $\Pi = \{V_1, \dots, V_k\}$ be a commanded partition with multiway expansion $\phi\Pi = \max_j \phi(V_j)$. Then the $(k-1)$ -th eigenvalue of the normalized Laplacian satisfies*

$$\mu_{k-1} \leq 2\phi\Pi. \quad (2)$$

Consequently, if Controller 1 (extended to k blocks) drives every $\phi(V_j) \leq \epsilon$, then $\mu_{k-1} \leq 2\epsilon$; the k sub-flocks are spectrally resolved once a gap $\mu_k - \mu_{k-1} > 0$ opens, and in the separated limit $\phi\Pi = 0$ the spectrum has a k -fold zero $\mu_0 = \dots = \mu_{k-1} = 0$ whose eigenspace is spanned by the indicators of the k connected components.

Proof. The indicator vectors $g_j = 1V_j, j = 1, \dots, k$, have disjoint supports, hence are linearly independent and span a k -dimensional subspace $U \subseteq \mathbb{R}^N$. Consider the generalized Rayleigh quotient

$R(v) = \frac{\langle v, Lv \rangle}{\langle v, Dv \rangle}$, whose critical values are the eigenvalues μ_i of $\mathcal{L}$ (equivalently the generalized eigenvalues of $L_V = \mu Dv$).

For $v = \sum_j c_j g_j \in U$, disjointness of supports gives

$$\langle v, Dv \rangle = \sum_j c_j^2 \text{vol}(V_j), \quad \langle v, Lv \rangle = \frac{1}{2} \sum_{a,b} a_{ab} (v_a - v_b)^2 = \sum_{j < j'} (c_j - c_{j'})^2 W(V_j, V_{j'}),$$

where $W(V_j, V_{j'}) = \sum_{a \in V_j, b \in V_{j'}} a_{ab}$ and only cross-group edges contribute. Using $(c_j - c_{j'})^2 \leq 2(c_j^2 + c_{j'}^2)$,

$$\langle v, Lv \rangle \leq 2 \sum_{j < j'} (c_j^2 + c_{j'}^2) W(V_j, V_{j'}) = 2 \sum_j c_j^2 \sum_{j' \neq j} W(V_j, V_{j'}) = 2 \sum_j c_j^2 |\partial V_j|.$$

Therefore, for every $v \in U \setminus \{0\}$,

$$R(v) = \frac{\langle v, Lv \rangle}{\langle v, Dv \rangle} \leq \frac{2 \sum_j c_j^2 |\partial V_j|}{\sum_j c_j^2 \text{vol}(V_j)} \leq 2 \max_j \frac{|\partial V_j|}{\text{vol}(V_j)} = 2\phi\Pi.$$

By the Courant–Fischer min–max characterization, $\mu_{k-1} = \min_{U' \subseteq U, \dim U' = k-1} \max_{v \in U'} R(v) \leq \max_{v \in U} R(v) \leq 2\phi\Pi$. The separated-limit statement is immediate: $\phi\Pi = 0$ means no cross-group edges, so G has (at least) k components with indicator null vectors, giving a k -fold zero eigenvalue.

Remark 5 (Two-sided reading). Theorem 4 is the test-set direction. The hard direction of the higher-order Cheeger inequality [10] states that some k -partition achieves expansion $O(k^2 \sqrt{\mu_{k-1}})$, so no commanded partition can beat the spectral k -way expansion by more than a poly(k) factor. The gap $\mu_k - \mu_{k-1}$ is therefore a two-sided monitor: small μ_{k-1} certifies that a good k -way split exists, and the opening gap certifies that the commanded one has been realized.

Hold and Per-group Tracking

Once separated, each sub-flock V_j tracks a commanded target q^* through an informed subset $\Gamma_j \subseteq V_j$ with feedback gains $c_i > 0, i \in \Gamma_j$. This is containment/pinning control [9, 1, 16] applied block-wise, and the Dirichlet certificate bounds its rate.

Proposition 6 (Per-group tracking certificate). In the hold phase, under hard pinning ($c_i \rightarrow \infty$ on Γ_j) the follower error on $V_j \setminus \Gamma_j$ decays at rate at least $\frac{1}{2} h_D(V_j \setminus \Gamma_j)^2$, where h_D is the Dirichlet Cheeger constant of the grounded induced subgraph $G[V_j]$ with Γ_j grounded. For finite gains the rate is $\lambda_{\min}(L(V_j) + P C_j)$, monotone in each c_i , interpolating from the weak-coupling value $\sum_{i \in \Gamma_j} c_i d_i$

$\text{vol}(V_j)$ up to the Dirichlet ceiling above.

Proof. Apply the Dirichlet certificate (1) and the soft-pinning interpolation of [13] to the induced subgraph $G[V_j]$, which is decoupled from the other blocks in the separated hold configuration ($\phi_{\Pi} = 0$).

Commanded Fusion

Fusion reverses the construction: group targets are driven to a common q^* until cross-block edges re-form, and the maintenance controller of [13] is applied to the whole graph.

Controller 7 (Fusion). $u_i = c(q^* - q_i) + \gamma \mathbb{K}[\lambda_1 < \epsilon] \nabla_{q_i} \lambda_1(G) + u_i^{\text{col}}$, with $\nabla_{q_i} \lambda_1(G)$ the global Fiedler-vector gradient.

Theorem 8 (Fusion re-cohesion certificate). Suppose Controller 7 restores a connected interaction graph with $h_N(q(t)) \geq h > 0$ for $t \geq t^*$. Then for $t \geq t^*$ the global velocity disagreement contracts at rate at least $\frac{1}{2} h^2$. (Neumann certificate). During the merge transient, with $L(q(t))$ continuously time-varying and weighted degrees in $[d, d]$, the disagreement obeys the uniform bound $\frac{\sqrt{2}}{2} \|\delta(t^*)\|_D^2 \exp(-\frac{d}{2} h^2 (t - t^*))$ of the companion time-varying theorem.

Proof. Post-merge, $h_N \geq h$ and the Neumann inequality (1) gives rate $\geq \frac{1}{2} h^2$. The transient bound is the companion continuously-time-varying cohesion certificate [13], whose hypotheses (bounded degrees, uniform Cheeger lower bound, C^1 weights) hold once cross-edges have formed.

The Switched Fission–Fusion System

The maneuver is a switched system with mode $\sigma(t) \in \{\text{split}, \text{hold}, \text{merge}\}$. Two design elements make it well-posed: a dwell time τ_d (each mode is held long enough to realize its certified contraction) and hysteresis thresholds $\epsilon_{\text{lo}} < \epsilon_{\text{hi}}$ on the spectral quantities (declare “separated” at $\mu_{k-1} \leq \epsilon_{\text{lo}}$, “merged” at $\lambda_1 \geq \epsilon_{\text{hi}}$), which prevents chattering at the crossings.

Proposition 9 (Well-posedness at the fission crossing). At the instant the flock separates, $\mu_{k-1} \rightarrow 0$ with multiplicity, the map $q \mapsto \mu_{k-1}$ loses differentiability and its (Clarke) gradient becomes setvalued. The group conductances $\phi(V_j)$, being ratios of sums of C^1 weights, remain differentiable in q wherever the weights are C^1 . Replacing the spectral surrogate $\nabla_{q_i} \lambda_1$ in Controller 1 by the conductance gradient $\nabla_{q_i} \phi(V_g(i))$ across the crossing yields a single-valued descent direction, so the controller is well-defined through the split.

Proof. $\phi(V_j) = |\partial V_j| / \text{vol}(V_j)$ is a quotient of finite sums of the C^1 weights $a_i \ell(\|q_i\|)$; where $\text{vol}(V_j) > 0$ the quotient rule gives a classical gradient. The eigenvalue μ_{k-1} is non-smooth exactly at the multiplicity point, where directional derivatives are governed by the Clarke subdifferential; substituting the smooth conductance gradient there avoids the set-valuedness while descending the same objective, since $\mu_{k-1} \leq 2\phi\Pi$ couples the two.

Theorem 10 (Switched fission–fusion stability). Assume dwell time $\tau_d > 0$ and uniform per-mode Cheeger lower bounds: $h_N(V_j) \geq h$ within every block during split and hold, and $h_N(G) \geq h$ globally during merge. Then over each dwell interval the active Lyapunov functional—the block disagreements $P_j \|\delta V_j\|^2$ in split/hold, the global disagreement $\|\delta\|_D^2$ in merge—contracts by a factor at most $e^{-h^2 \tau_d}$ (up to the condition-number prefactor of Theorem 8). With hysteresis $\epsilon_{\text{lo}} < \epsilon_{\text{hi}}$ the schedule aggregated $\rightarrow$ fission $\rightarrow$ hold $\rightarrow$ fusion $\rightarrow$ aggregated admits finitely many

switches, every phase is certified, and the flock returns to a cohesive state.

Proof. On each dwell interval the mode is fixed and the relevant certificate of Theorems 2 and 8 and Proposition 6 applies, giving the stated per-interval contraction by piecewise Gronwall integration of the corresponding Rayleigh inequality; this is the dwell-time / jointly-connected composition of the companion switching result [13, 19, 8]. Hysteresis separates the “separated” and “merged” declaration thresholds, so each transition fires at most once per cycle, bounding the number of switches.

Numerical Experiments

A commanded three-way disaggregate–hold–re-aggregate cycle We simulate a commanded $k=3$ disaggregate–hold–re-aggregate cycle with $N = 30$ agents (ten per group) in the plane, Gaussian kernel $a_{ij} = \exp(-\|q_{ij}\|^2/2\ell^2)$ truncated at sensing radius r , integrated with a damped double integrator. The schedule is aggregated $\rightarrow$ fission (groups steered to a triangle of targets, with block-connectivity maintenance) $\rightarrow$ hold $\rightarrow$ fusion (targets returned to a common center, with global maintenance). At each step we form $L(q)$ and record the global algebraic connectivity, each block’s $\lambda_1(V_j)$, the group conductances $\phi(V_j)$, and the low spectrum.

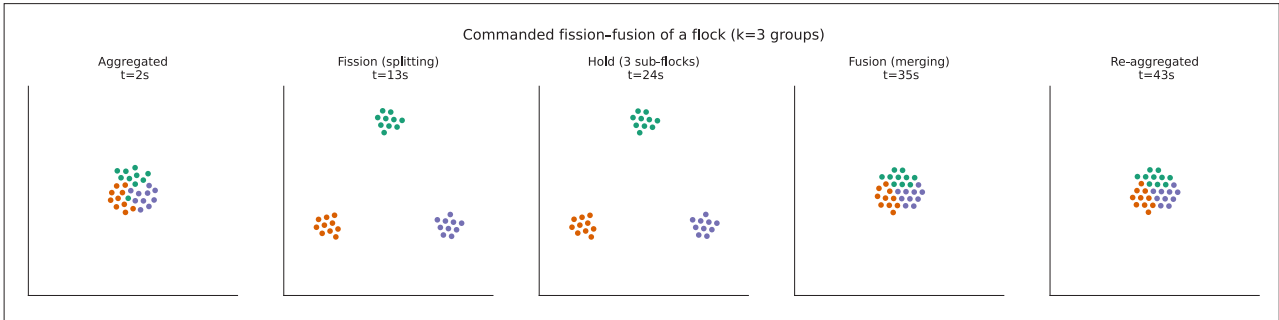


Figure 1: Commanded Fission Fusion of a Flock ($k = 3$). The Swarm Starts as one Connected Flock (three spatially coherent groups), Splits on Command to a Triangle of Targets, holds as Three Independent Sub-Flocks, and Re-Aggregates on the Fusion Command

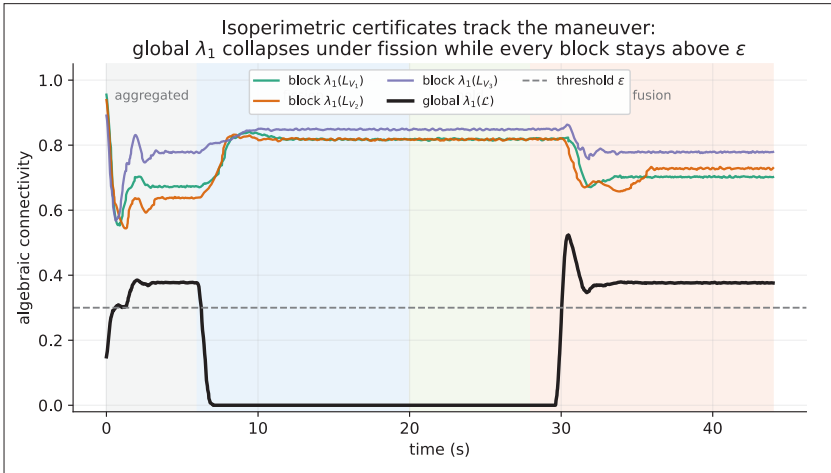


Figure 2: The Certificates Track the Maneuver. The Global Algebraic Connectivity $\lambda_1(L)$ (black) Collapses to Zero During Fission and is Restored on the Fusion Command, While Every Block Connectivity $\lambda_1(L_{V_j})$ (colored) Stays above the Threshold ϵ Throughout Split and Hold: Each Sub-Flock Remains a Coherent Flock Even as the Whole Disaggregates, Exactly as Theorem 2 Requires. Phases are Shaded

Figure 2 is the central result: through the entire fission and hold the minimum block connectivity is 0.639 (well above $\epsilon = 0.30$), while the global λ_1 reaches 0 (three components) and then recovers 7 to 0.376 after the fusion command. Figure 3 shows the multiway signature of Theorem 4: during hold the two lowest nontrivial eigenvalues $\mu_1, \mu_2 \rightarrow 0$ while the gap eigenvalue μ_3 stays at 0.81, and every group conductance $\phi(V_j) \rightarrow 0$. Figure 4 shows each group tracking its commanded target.

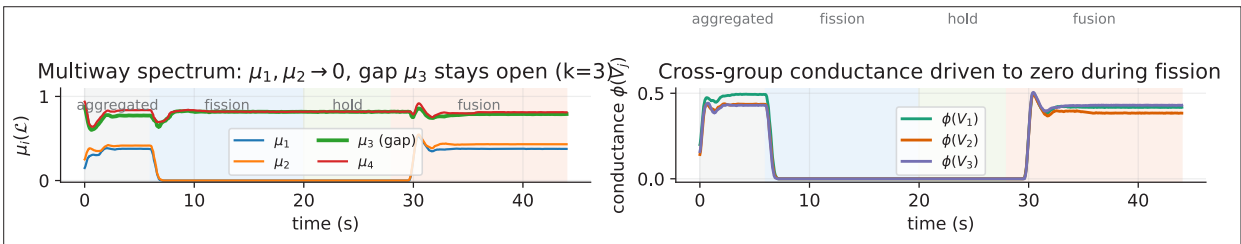


Figure 3: LEFT: The Low Spectrum. Under Fission the $k-1 = 2$ Lowest Nontrivial Eigenvalues μ_1, μ_2 Fall to Zero While the Gap μ_3 Stays Open the Higher-Order Cheeger Signature of a Clean 3-Way Split (Theorem 4, Remark 5). Right: the Group Conductances $\phi(V_j)$ Driven to Zero, Upper-Bounding μ_{k-1} via (3)

Minimum-Effort Disaggregation

Proposition 3 predicts that commanded splitting along the Fiedler/Cheeger cut severs the least cross-cut weight. We test this over 50 random connected flocks ($N = 40$): for each we form the balanced Fiedler cut (thresholding the Fiedler vector at its median) and compare its split effort $\mathcal{E}(S) = |\partial S|$ against 200 arbitrary balanced cuts of the same size. Figure 5 reports the outcome: an arbitrary balanced cut costs on average $2.83\times$ the Fiedler cut's effort, and even the best of 200 random cuts costs $2.14\times$; in every one of the 50 flocks the Fiedler cut is no worse than the *best* random cut. Commanding the split along the Fiedler cut is thus the minimum-effort disaggregation, as Proposition 3 asserts.

The Multiway Certificate Across k

We validate the higher-order bound $\mu_{k-1} \leq 2\phi_{\Pi}$ of Theorem 4 directly, for $k \in \{2, 3, 4, 5\}$. For each k we generate 50 configurations of k spatially coherent groups at separations ranging from overlapping to well separated, and for each commanded k -partition record the $(k-1)$ -th eigenvalue μ_{k-1} and the bound $2\phi_{\Pi} = 2 \max_i \phi(V_i)$. Figure 6 plots one against the other: across all 200 partitions every point lies below the certificate line $\mu_{k-1} = 2\phi_{\Pi}$, confirming (3). The bound is not merely valid but tight, with median slack $2\phi_{\Pi}/\mu_{k-1}$ between 1.3 and 1.9 (Table 1); the slack grows modestly with k , consistent with the poly(k) looseness of the higher-order inequality noted in Remark 5.

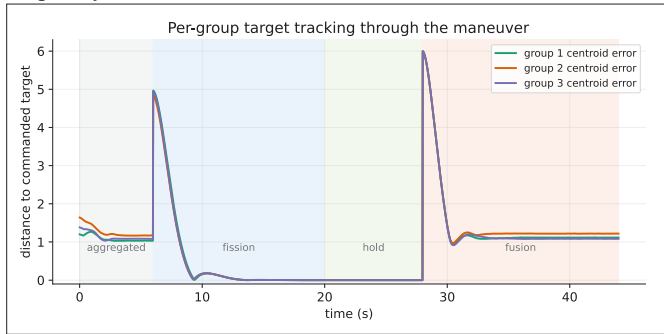


Figure 4: Per-Group Centroid Tracking Error to the Commanded Targets Through the Maneuver (Proposition 6): Each Sub-Flock Acquires Its Separation Target During Fission, Holds It, And Returns To The Common Target During Fusion

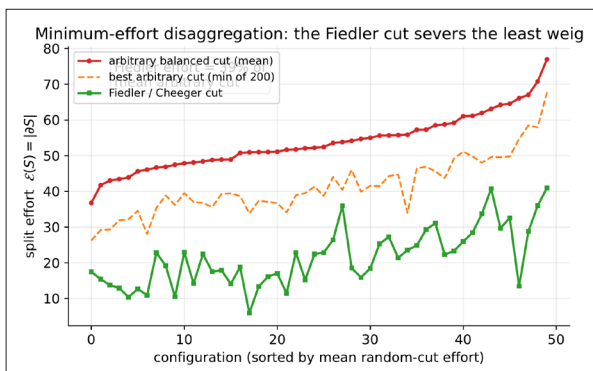


Figure 5: Minimum Effort Disaggregation (proposition 3). Split Effort $\mathcal{E}(S) = |\partial S|$ of the Fiedler/Cheeger Cut (green) Versus the Best of 200 Random Balanced Cuts (orange, dashed) and their Mean (red), Over 50 Random Flocks Sorted by Mean Random Cut Effort. The Fiedler Cut Lies at or Below the Best Random Cut in Every Configuration

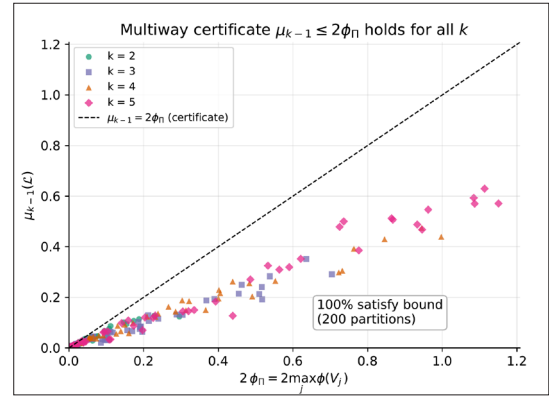


Figure 6: Multiway Fission Certificate $\mu_{k-1} \leq 2\phi_{\Pi}$ (Theorem 4) over 200 Commanded Partitions, $k \in \{2, 3, 4, 5\}$. Every Point Lies Below the Diagonal, and Both Quantities Vanish Together as a Split Completes (lower left). The Bound Holds for all k

Table 1: Multiway Certificate Slack by k . The Bound $\mu_{k-1} \leq 2\phi_{\Pi}$ Holds in all 200 Partitions (100%); the Median Slack Stays Between 1.3 and 1.9

k	artitions	median slack $2\phi_{\Pi}/\mu_{k-1}$
2	50	1.32
3	50	1.88
4	50	1.68
5	50	1.77

A Four-way Commanded Split

To show the framework is not special to $k = 3$, we repeat the commanded maneuver with $k = 4$ groups ($N = 48$) steered to the vertices of a square (Figure 7). The certificates behave identically: through fission and hold the minimum block connectivity is 0.662 (above $\epsilon = 0.30$), while the three lowest nontrivial eigenvalues $\mu_1, \mu_2, \mu_3 \rightarrow 0$ and the gap μ_4 holds at 0.693 (Figure 8)—the higher-order Cheeger signature of a clean four-way disaggregation predicted by Theorem 4.

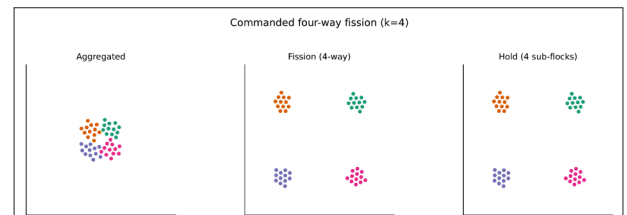


Figure 7: Commanded Four-Way Fission ($k = 4, N = 48$): The Aggregated Flock, The Four-Way Split to a Square of Targets, and the Hold as Four Independent Sub-Flocks

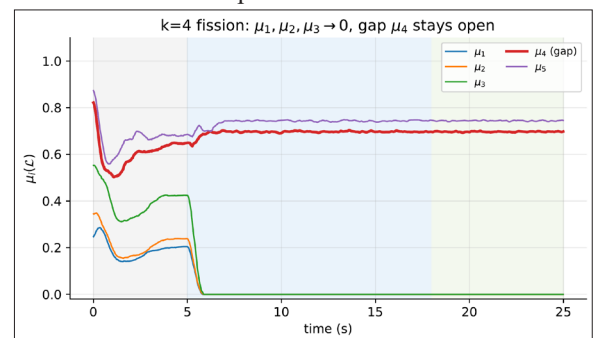


Figure 8: Low Spectrum of the Four-Way Split. The $k-1 = 3$ Lowest Nontrivial Eigenvalues μ_1, μ_2, μ_3 Fall to Zero Under Fission While the Gap μ_4 Stays Open, Certifying a Clean Four-Way Resolution (Theo-rem 4)

Conclusion

We have given a certificate-based framework for commanded fission–fusion of flocks. The governing idea is that organized disaggregation is the flocking fragmentation criterion used as an objective: the isoperimetric gradient that maintains connectivity, descended block-wise, splits the flock along a commanded cut while each sub-flock retains a Neumann-certified cohesion rate. The two-way certificate (Theorem 2), the higher-order multiway certificate (Theorem 4), the per-group Dirichlet tracking certificate (Proposition 6), and the fusion certificate (Theorem 8) assemble into a dwelltime switched controller whose every phase is guaranteed, with the eigenvalue crossing at the split handled by handoff to the conductance gradient (Proposition 9). Numerically, a commanded $k=3$ cycle confirms every bound.

Several directions remain. The multiway controller inherits the $\text{poly}(k)$ looseness of the hard higher-order Cheeger direction; sharper k -way guarantees, decentralized estimation of the first k eigenpairs during the maneuver, collision-safe splitting retaining the full nonlinear separation potential, and adversarial fission (splitting to evade a pursuer, as in the bio-inspired treatments [24, 6]) are natural next steps. A mean-field account of commanded k -way splitting, in which the discrete multiway Cheeger constant converges to a continuum partition functional, would connect this control problem to the continuum isoperimetry conjectured in the companion work.

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